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Oscillators with Chaotic Behavior: An Illustration of a Theorem by Shil'nikov

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Using an explicit one-parameter family of differential equations describing oscillators with feedback effects, we prove the existence of values of the parameters such that there exist infinitely many unstable periodic orbits of saddle type. The proof relies on a theorem by Shil'nikov which we propose as an explanation for the origin and structure of the chaotic behavior displayed by many well-known third-order differential systems.

KEY WORDS: Strange attractor; homoclinic connection; saddle focus; oscillator.

1. INTRODUCTION

For many years, a lot of effort has been devoted to the study of stochasticity inherent in the solutions of ordinary differential equations. From the physical point of view, such equations either describe particularly simple (generally electromechanical) systems,[1-7] or arise as simplified models when investigating more complicated systems naturally described, e.g., by partial differential equations (a historical example is the well-known Lorenz model for convection[8]). Most current interest in such topics corresponds to an interest in understanding the mechanisms of the onset of turbulence in physical, chemical, biological, etc. systems.[9] Unfortunately, as soon as one intends to define turbulence in the frame of an appropriate statistical mechanics, one is faced with very profound and delicate mathematical problems such as the existence of strange attractors and invariant measures

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with strong ergodic properties.[10] In particular, as far as reasonably realistic models are considered like differential systems given by explicit algebraic equations, this objective is still far beyond our possibilities, although there seems to be numerical evidence for the existence of actual strange attractors in many of these systems.

A fundamental step in the investigation of stochasticity in such differential equations consists in proving the existence of infinitely many isolated periodic orbits. However, it seems doubtful if not hopeless to get general theorems giving definitive answers to this last problem under analytically computable conditions. The main purpose of this paper is to give convincing arguments that the origin and structure of chaotic behavior displayed by many one-parameter families of third-order differential equations can be understood in the light of the following theorem, which is a slight modification of an old result by Shil'nikov[11] (see also Refs. 12–16 for analog results in four-dimensional Hamiltonian systems):

Theorem 1. Consider the system

$$\begin{aligned} \dot{x} &= \rho x - \omega y + P(x, y, z) \\ \dot{y} &= \omega x + \rho y + Q(x, y, z) \\ \dot{z} &= \lambda z + R(x, y, z) \end{aligned} \tag{1}$$

where P, Q, R are C^r functions ($1 \leqslant r \leqslant \infty$) vanishing together with their first derivative at $O = (0, 0, 0)$.[3] Let us assume that one of the orbits, denoted by Γ_0, is asymptotic to O as $t \to \pm\infty$, being bounded away from any other singularity (Γ_0 is then a homoclinic connection). Then if

$$\lambda > -\rho > 0 \tag{2}$$

every neighborhood of the unstable orbit Γ_0 contains a denumerable set of unstable periodic solutions of saddle type.

As a first argument we define in Section 2, a solvable model for which on the one hand one can observe numerically chaotic behavior and on the other hand one can prove that the previous theorem applies. The second part of our argumentation consists of Section 3, in the numerical investigation of three models extracted from the literature. We conclude with some remarks in Section 4.

2. A SOLVABLE MODEL

Clearly, all conditions in Theorem 1 are easily computable, except the existence of the homoclinic connection. In this section, we will exhibit

[3] Using Poincaré's terminology, the origin is then a saddle focus.

one-(free)parameter families of differential systems where this difficulty can be overcome. These families arise as specifications of the class of oscillators with feedback effects introduced in Ref. 17 and whose equations may be written as

$$\begin{aligned} &\ddot{x} + \beta\dot{x} + x = \eta \\ &\dot{\eta} = f_\mu(x) \end{aligned} \qquad \text{or} \qquad \begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -y - \beta z + f_\mu(x) \end{aligned} \tag{3}$$

where $\beta > 0$ is the dissipation and f_μ a one-parameter real-valued function. Various sequences of bifurcations leading to chaotic behavior can be obtained with (3) by appropriate choices of $f_\mu(x)$. Moreover, with the particular choice of piecewise linear functions f_μ,

$$\begin{aligned} f_\mu(x) &= 1 + ax \qquad \text{if} \quad x \leqslant 0 \\ f_\mu(x) &= 1 - \mu x \qquad \text{if} \quad x \geqslant 0 \end{aligned} \tag{4}$$

we can understand the origin of the numerically observed stochasticity since we can find values of the parameters such that the conditions of the previous theorem are satisfied,[4] namely, we can find β, a, and μ such that an orbit Γ_0 leaves the fixed point $A(-1/a, 0, 0)$ and returns to it as $t \to +\infty$, the eigenvalues of the Jacobian matrix at A satisfying condition (2) (of course, it is the piecewise-linear character of f_μ which makes the calculations easy). Let us sketch the main steps of the proof of the existence of such a parameter set.

2.1. Step 1: Algebraic Considerations

First, we must remark that if a and μ are positive, one has two equilibria for system (3); one $A(-1/a, 0, 0)$ is in the half-space $x \leqslant 0$, the other one $B(1/\mu, 0, 0)$ in the half-space $x \geqslant 0$. Let us denote by λ and $\rho \pm i\omega$ the eigenvalues of the Jacobian matrix at A, and by L and $R \pm i\Omega$ those corresponding to B. For the sake of simplicity, (ρ, ω, R) are chosen as parameters instead of (β, a, μ). Then λ, β, and a are given by

$$\begin{aligned} \lambda &= (1 - \rho^2 - \omega^2)/2\rho \\ \beta &= -(\lambda + 2\rho) \end{aligned} \tag{5}$$

and

$$a = \lambda(\rho^2 + \omega^2)$$

[4] Of course, with system (3) and $f_\mu(x)$ given by Eq. (4), one has not the smoothness condition required in Theorem 1. The proof[13] for the C^r case ($1 \leqslant r \leqslant \infty$) extends however trivially to the problem considered here.

Since the point A will play the role of the origin in the above theorem, condition (2) is thus equivalent to

$$\rho^2 + 1 < \omega^2 < 3\rho^2 + 1 \tag{6}$$

Similarly, the expressions for L, Ω, and μ are

$$\begin{aligned} L &= (1 + 3\rho^2 - \omega^2)/2\rho - 2R \\ \Omega^2 &= 1 + 3R^2 - R(1 + 3\rho^2 - \omega^2)/\rho \end{aligned} \tag{7}$$

and

$$\mu = (R^2 + \Omega^2)(R^2 + \Omega^2 - 1)/2R$$

Indeed, in order to satisfy Eqs. (2) or (6) together with the existence of Γ_0, we choose *a priori* ρ and ω and let R vary as a free parameter.

2.2. Step 2: Invariant Manifolds of A

The invariant manifolds of A coincide with affine manifolds in the neighborhood of this equilibrium. The unstable manifold corresponding to the eigenvalue λ intersects the $x = 0$ plane at the point $M(0, \lambda/a, \lambda^2/a)$. The stable manifold is a plane in a neighborhood of A. This plane Π^- intersects the $x = 0$ plane along the line D^- whose equation is given by

$$x = 0, \qquad z = 2\rho\left[y + 1/(\rho^2 + \omega^2 - 1)\right] \tag{8}$$

All points of D^- are not necessarily points of the stable manifold of A. The following remark which gives a sufficient condition for a point of D^- to belong effectively to this manifold will be useful for the computation of the orbit Γ_0.

Remark I. If $N \in D^-$ is such that the orbit which originates from N at $t = 0$ is included in the $x \leqslant 0$ half-space for all $t > 0$, then N belongs to the stable manifold of A.

2.3. Step 3: Existence of Γ_0

We are looking for a homoclinic orbit Γ_0 which consists of three parts.

Part one: the line segment AM belonging to the unstable manifold of A with the point M lying in the $x = 0$ plane.

Part two: MN defined as the piece of the orbit which originates from M at $t = 0$ and reaches the point N on the line D^- at a time $t < 2\pi/\Omega$ (this last condition is not essential but allows easier computations[5]).

[5] In such a problem, one can expect for infinitely many values of the parameter μ (a and β adequately fixed) such that a homoclinic orbit Γ_0 exists: in Section 3 we show examples of Γ_0 with different structures obtained with Eq. (3) and an analytic $f_\mu(x)$.

Part three: the orbit issued from N which, when the condition in Remark I is fulfilled, admits A as ω-limit point.

The construction of part one of Γ_0 is trivial. In order to construct the second part, let

$$\begin{bmatrix} x_R(t) \\ y_R(t) \\ z_R(t) \end{bmatrix} = \begin{bmatrix} (u\cos\Omega t + v\sin\Omega t)\cdot \exp Rt + w\exp Lt + 1/\mu \\ \dot{x}_R(t) \\ \ddot{x}_R(t) \end{bmatrix} \tag{9}$$

be the solution of Eq. (3) with $f_\mu(x) \equiv 1 - \mu x$ (for all x) and initial condition M at $t = 0$. Let furthermore P_1 (resp. P_2) be the half-space delimited by Π^- and containing (resp. not containing) M. Using adequate ρ and ω, one can find $0 < R_1 < R_2$ and $0 < t_1 < t_2 < 2\pi/\Omega$ such that

$$\begin{cases} x_R(t_1) > 0 & \text{for} \quad R \in [R_1, R_2] \\ x_R(t_2) < 0 & \text{for} \quad R \in [R_1, R_2] \\ x_{R_1}(t) \in P_1 & \text{for} \quad t \in [t_1, t_2] \\ x_{R_2}(t) \in P_2 & \text{for} \quad t \in [t_1, t_2] \end{cases} \tag{10}$$

Then a simple continuity argument ensures the existence of $R^* \in]R_1, R_2[$ and $t^* \in]t_1, t_2[$ such that $N(x_{R^*}(t^*), y_{R^*}(t^*), z_{R^*}(t^*))$ belongs to D^-.

To end the proof, we must check that N satisfies the condition of Remark I, namely, that the orbit issued from the point N stays in the $x \leqslant 0$ half-space. N necessarily belongs to the part of the line D^- which lies between the plane Π_R^+ containing the local unstable manifold of $B_R(1/\mu, 0, 0)$ and the plane parallel to Π_R^+ through M; denote by I_R and J_R the intersection points of D^- with these planes and let

$$d_1 = \sup_{R \in [R_1, R_2]} \left(\max(d(A, I_R), d(A, J_R))\right) \tag{11}$$

where d stands for the usual distance. Then, if d_0 denotes the distance from A to D^-, it suffices to prove that wherever the point N is on the segment $\bigcup_{R \in [R_1, R_2]} [I_R, J_R]$, the orbit issued from N lies, after half a revolution around A, at a distance from this point which is lower than d_0. Clearly, it is enough to take ρ, ω, R_1, and R_2 such that

$$d_0 > d_1 \exp(\rho\pi/\omega) \tag{12}$$

As an example, the particular parameter set $\rho = -0.4$, $\omega = 1.1$, $R_1 = 0.39$, and $R_2 = 0.4$ satisfies all the required conditions; thus we can claim that *there exist families of differential equations given by Eqs.* (3) *and* (4) *for which Theorem* 1 *applies.*

In fact, for $\rho = -0.4$ and $\omega = 1.1$, the computer furnishes $R_1' = 0.3982$ and $R_2' = 0.3983$ as better bounds for R^* such that Γ_0 exists: the unstable manifold of A for both cases is represented in Figs. 1a and 1b. Keeping ρ and ω fixed but moving R to the value $R = 0.1740$, we observe numerically a "strange attractor" (Fig. 1c) whose structure reflects the existence of Γ_0

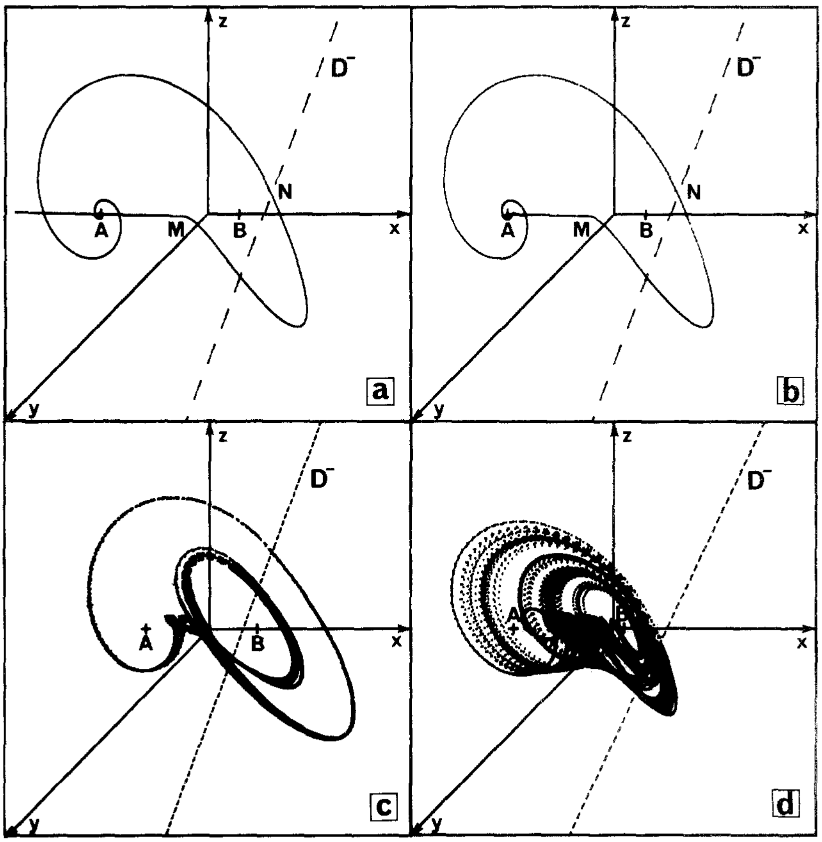

Fig. 1. The oscillators (3) with piecewise linear $f_\mu(x)$ given by Eq. (4) and parameters values $\rho = -0.4$, $\omega = 1.1$ ($a = 0.633625$ and $\beta = 0.3375$). The half of the unstable manifold of A going through M (as defined in the main text) diverges along the part of the unstable manifold of A which does not intersect the $x = 0$ plane for $R = 0.3983$ (a) and comes back very close to M for $R = 0.3982$ (b). We have also plotted the reference frame and the line D^-. For $R = 0.1740$, a "strange attractor" is numerically displayed whose structure reflects the existence (for a larger value of R: $0.3982 < R^* < 0.3983$) of a homoclinic orbit Γ_0 satisfying the hypothesis of Theorem 1. Full (dotted) line corresponds to the $x \geqslant 0$ ($x \leqslant 0$) half-space. For parameter values $\rho = -0.27$ and $\omega = 1.018$ ($a = 0.224635$, $\beta = 0.3375$), condition (2) in Theorem 1 is no longer fulfilled. (d) shows that chaotic behavior may still occur in this case where $-\rho > \lambda > 0$.

for a larger value of R. It should be noticed that even when $-\rho > \lambda > 0$, chaotic behavior may still occur while stable periodic orbits exist for values of the parameters in the neighborhood of values where Γ_0 arises. A way to satisfy these last inequalities on the characteristic exponents evaluated at A is to increase the dissipation β, a and μ being fixed. For still larger values of β, the equilibrium A no longer remains a saddle focus so that one can no more expect to observe chaotic behavior (see Section 4). This corroborates previous remarks reported in Ref. 17 where no stochasticity was found with the oscillators (3) for "too strong" dissipation values.

3. NUMERICAL EXAMPLES

In this section we will pursue our argumentation in examining some three-dimensional differential systems which display "strange attractors" with the same geometrical structure as observed in Fig. 1c with our solvable model. Convincing numerical data make clear that the structure of these objects still relies on the existence of homoclinic orbits through a saddle focus satisfying condition (2).

3.1. Example 1: The Oscillators with Feedback Effect as Defined by Eq. (3) but with Analytical $f_\mu(x)$[17]

Figure 2a–c gives numerical evidence for the occurrence, with $f_\mu(x) = \mu x(1-x)$ of a succession of unstable homoclinic orbits, the simplest one being similar to the one previously investigated with the piecewise linear function f_μ. A typical "attractor" whose structure reflects the existence of these homoclinic orbits is represented in Fig. 2d.

3.2. Example 2: A "Prototype" Equation Proposed by Rössler[18–20]

The simple differential system

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= bx - cz + xz \end{aligned} \tag{13}$$

has been devised by Rössler in order to give examples of what he calls, respectively, "screw type" (Figs. 1, 2, and 3) and "spiral type" (Fig. 4) strange attractors. Figure 3a–c (to be compared with Figs. 1a–c) shows that the "screw-type" attractor reveals that one is "close" to a situation where Theorem 1 applies (it seems that Rössler had already the feeling of such a phenomenon). "Spiral-type" attractors will be discussed in the conclusion.

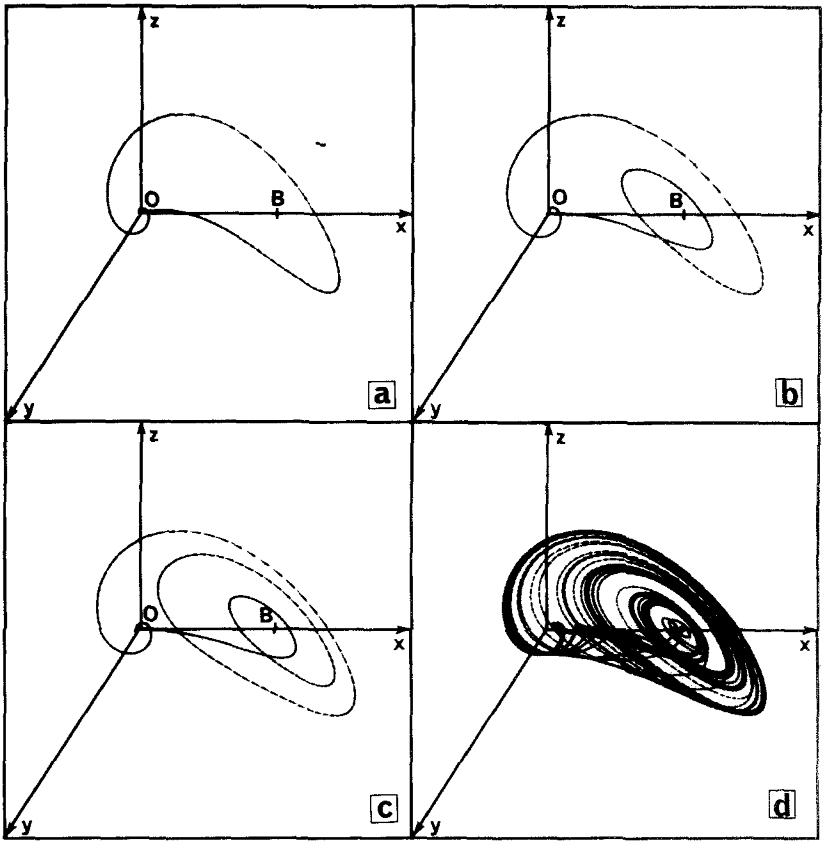

Fig. 2. The oscillators (3) with $f_\mu(x) = \mu x(1-x)$ and $\beta = 0.4$. As mentioned in footnote 5, we have convincing numerical data that confirm the existence of a succession of unstable homoclinic orbits biasymptotic to the origin and satisfying Theorem 1. The simplest ones are represented in (a) $\mu = 1.6064$, (b) $\mu = 1.0232$, (c) $\mu = 0.9148$. A characteristic "strange attractor" is observed for $\mu = 0.872$ (d). The structure of this attractor strongly suggests that one is also not too far from values of the parameters such that the second equilibrium $B(1,0,0)$ plays the role of the origin in Theorem 1, the time t being replaced by $-t$.

3.3. Example 3: A Three-Mode Model for the Development of a Modulation Instability in Nonequilibrium Media

In Ref. 21, Fabrikant and Rabinovich discuss the system

$$\begin{aligned} \dot{x} &= y(z-1+x^2)+\gamma x \\ \dot{y} &= x(3z+1-x^2)+\gamma y \\ \dot{z} &= -2z(\nu+xy) \end{aligned} \tag{14}$$

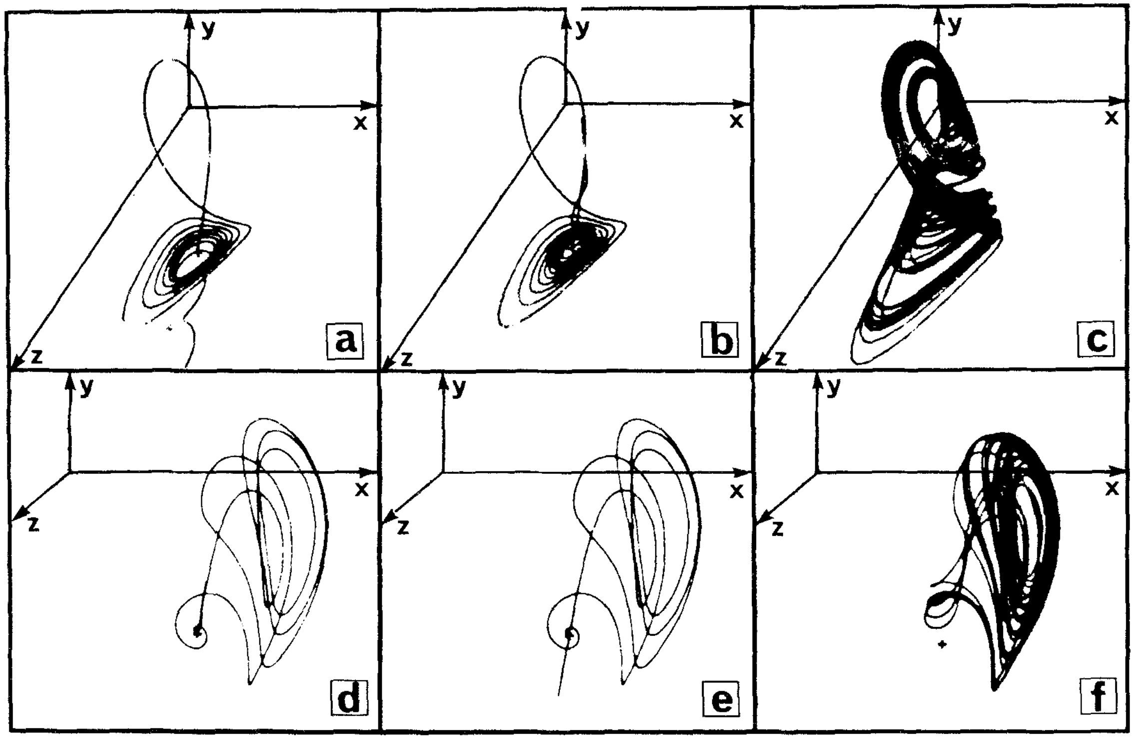

Fig. 3. "Screw-type" attractors in the Rössler's prototype Eq. (13) and the three-mode model (14) of Fabrikant and Rabinovich. These attractors presented in (c) and (f), respectively, seem to be well understood in terms of Theorem 1 as suggested, respectively, by (a–b) and (d–e) (to be compared with Fig. 1a–b). Rössler's equation calculations for $b = 0.4$, $c = 4.5$; $a = 0.58713$ (a), 0.58712 (b), 0.5 (c). Fabrikant–Rabinovich three-mode model calculations for $\nu = 1.1$; $\gamma = 0.94775$ (d), 0.94770 (e), 0.87 (f).

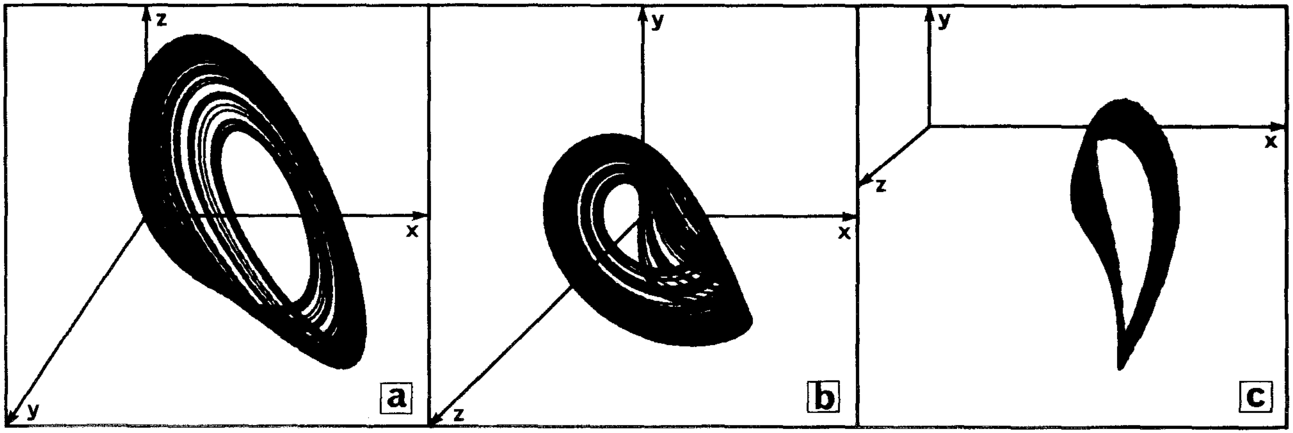

Fig. 4. "Spiral-type" attractors displayed by (a) the oscillators (3) with $f_\mu(x) = \mu x(1 - x)$, $\beta = 0.4$ and $\mu = 0.8$; (b) the Rössler's Eq. (13) with $a = 0.33$, $b = 0.4$, $c = 4.5$; (c) the Fabrikant–Rabinovich model (14) with $\nu = 1.1$, $\gamma = 0.77$.

which is a three-mode model for the description of a stochastic modulation of waves in one-dimensional media. Figure 3f represents a "strange attractor" exhibited by Eq. (14) for the parameters set $\nu = 1.1, \gamma = 0.87$. Figure 3d–e gives numerical evidences that a homoclinic orbit Γ_0 exists for a value of γ slightly larger and that Theorem 1 is again quite appropriate to explain the observed chaotic behavior.

Obviously, we do not pretend to give an exhaustive review of the three-dimensional differential systems displaying chaotic behavior interpretable in terms of Theorem 1. Nevertheless, we think that the three previous examples show significantly the efficiency of this theorem seen as a criterion which allows one to distinguish systems generating stochasticity.

4. CONCLUSION

To conclude, let us remark that in none of these examples, chaos is to be observed for values of the parameters where Γ_0 is supposed to exist; this interesting situation requires special conditions on the system as e.g., the invariance under the transformation $(x, y, z) \to (-x, -y, -z)$ as proposed in Ref. 3. Indeed, when symmetries (or "nearby" symmetries) are invoked, even homoclinic orbits through equilibria with real characteristic exponents can give rise to chaotic behavior as described, e.g., in Refs. 8, 14, and 22–27. It is characteristic for distinguished systems of the kind presented here that symmetry properties are not necessary to make the existence of stochasticity plausible.

Let us end with a somewhat more technical remark: the evolution from "screw-type" (Figs. 1, 2, and 3) to "spiral-type" (Fig. 4) attractors can be understood using the proof of Theorem 1.[11–14] More precisely, when Γ_0 exists, there is a horseshoe with infinitely many branches in a typical Poincaré first return map for the flow; the number of branches decreases when we move the parameters away from values where a homoclinic orbit exists.[13,28] This evolution explains in turn that well-known phenomena such as cascades of subharmonic bifurcations,[17,28–35] or intermittency[36–38] naturally arise when varying a parameter in such systems.

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